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APPLICATION OF PROGRESSION THEORY WHEN CALCULATING THE LIMIT OF A SEQUENCE OF NUMBERS**Jura Oramov**Associate Professor of the Department of General
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Annotation: The concept of a sequence of numbers, its limit, is one of the most important basic concepts of mathematical analysis. There are many methods for calculating the limit of the sequence of numbers. In this article, the application of progression theory in calculating the limit of a sequence of numbers is investigated on the basis of concrete examples.

Keywords: sequence of numbers, concept of limit, arithmetic progression, geometric progression, infinitely decreasing progression, private sums

Introduction.

The concept of a sequence of numbers and its limit is one of the most important fundamental concepts of mathematical analysis and occupies a special place. The theory of limits forms not only the basis of theoretical mathematics but is also widely applied in many fields such as physics, economics, engineering, and information technologies. In the process of determining the limit of a numerical sequence, various mathematical methods are used, among which the theory of progressions plays an important role.

Arithmetic and geometric progressions are effective tools for studying the regularities of sequences and analyzing their convergence or divergence properties. In particular, when calculating the limits of complex sequences, the formulas and properties of progressions help to simplify the problem. Therefore, studying the applications of progression theory in calculating limits is one of the important directions of mathematical analysis.

This article presents the possibilities of using arithmetic and geometric progressions in calculating limits of numerical sequences, their theoretical foundations, and their analysis through practical examples. It also examines effective methods of determining limits using progressions and reveals their significance in developing mathematical thinking.

Literature review on the topic.

The literature used in covering the topic “Applications of the theory of progressions in calculating the limits of numerical sequences” plays an important role in revealing both the theoretical and practical aspects of this subject. In the work “Problems in Mathematical Analysis” by N.Ya. Vilenkin and co-authors, various problems of mathematical analysis extensively present methods for calculating the limits of sequences, and examples related to progressions help in gaining a deeper understanding of the topic. In S.I. Novoselov’s book “Special Course in Elementary Algebra”, the properties of progressions and their role in determining limits of sequences are explained based on algebraic methods. In the textbook “Mathematical Analysis” by T. Azlarov and X. Mansurov, the concepts of sequences and limits are consistently presented, and methods of calculating limits using arithmetic and geometric progressions are demonstrated through theoretical and practical examples. In N.S. Glagolov and others’ work “Mathematics”, concepts related to progressions are explained in a simple and clear manner, and their importance in finding limits of sequences is highlighted.

Research methodology. A numerical sequence is an infinite set of consecutively ordered numbers $x_1, x_2, x_3, \dots, x_n, \dots$, denoted by $\{x_n\}$, which is defined by some rule that determines $x_n = f(n)$ as a function of the natural numbers.

One of the important operations studied in mathematical analysis is the operation of taking the limit. We give the definition of the limit of a sequence of numbers: If for every $\varepsilon > 0$ there exists a natural number $n_0 > N$ such that for all natural numbers satisfying $n > n_0$,

$$|x_n - a| < \varepsilon$$

then the number a is called the limit of the sequence $\{x_n\}$, and for the limit we use the notation

$$\lim_{n \rightarrow \infty} x_n = a \text{ or } x_n \rightarrow a \text{ as } n \rightarrow \infty.$$

In the article, since we use applications of the theory of progressions in calculating the limits of numerical sequences, we present their relevant properties: **1°. The formula for any term of an arithmetic progression:**

$$a_n = a_1 + (n-1)d, n \in \mathbb{N} \quad (a_1 - \text{the first term, } d - \text{difference})$$

2°. Sum of the first n terms of an arithmetic progression:

$$S_n = \frac{a_1 + a_n}{2} \cdot n$$

3°. Formula for any term of a geometric progression:

$$b_n = b_1 \cdot q^{n-1}, n \in \mathbb{N} \quad (b_1 - \text{first term, } q - \text{ratio of the progression})$$

4°. Sum of the first n terms of a geometric progression:

$$S_n = \frac{b_1(1-q^n)}{1-q}, (q \neq 1)$$

5°. Sum of an infinite decreasing geometric progression:

$$S = \frac{b_1}{1-q}, (|q| < 1)$$

We present examples of problems solved using the theory of progressions in calculating the limits of numerical sequences.

Example 1. Compute the limit of the sequence

$$x_n = \frac{1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}}{1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n}}, n \in \mathbb{N}.$$

Solution: Since the sequence $\{x_n\}$ is composed of a finite geometric progression with $b_1 = 1$, $q = \frac{1}{2}$, and $b_{n+1} = \frac{1}{2^n}$, we have:

$$S_{n+1} = \frac{b_1(1-q^{n+1})}{1-q} = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2 \left(1 - \frac{1}{2^{n+1}} \right).$$

The sequence $\{x_n\}$ consists of the sum of a finite geometric progression with $b_1 = 1$, $q = \frac{1}{3}$, and $b_{n+1} = \frac{1}{3^n}$.

$$\sigma_{n+1} = \frac{b_1(1-q^{n+1})}{1-q} = \frac{1 - \frac{1}{3^{n+1}}}{1 - \frac{1}{3}} = \frac{3}{2} \left(1 - \frac{1}{3^{n+1}} \right).$$

Then,

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \frac{S_{n+1}}{\sigma_{n+1}} = \frac{\lim_{n \rightarrow \infty} 2 \left(1 - \frac{1}{2^{n+1}}\right)}{\lim_{n \rightarrow \infty} \frac{3}{2} \left(1 - \frac{1}{3^{n+1}}\right)} = \frac{2 \left(\lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{2^{n+1}}\right)}{\frac{3}{2} \left(\lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{3^{n+1}}\right)} = \frac{2(1-0)}{\frac{3}{2}(1-0)} = \frac{4}{3}$$

Thus, the answer is $\frac{4}{3}$

Example 2. If $x_n = \frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n-1}{n^2}$, find $\lim_{n \rightarrow \infty} x_n$.

Solution: We write x_n as follows:

$$x_n = \frac{1+2+3+\dots+n-2+n-1}{n^2}.$$

It is obvious that the terms in the numerator form an arithmetic progression, where $a_1=1$, $d=1$, and $a_{n-1}=n-1$. Using the formula for the sum of a finite arithmetic progression:

$$S_{n-1} = 1+2+3+\dots+(n-1) = \frac{[1+(n-1)](n-1)}{2} = \frac{n(n-1)}{2}$$

Then for x_n we obtain:

$$x_n = \frac{n(n-1)}{2n^2} = \frac{n-1}{2n}$$

As a result,

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \frac{n-1}{2n} = \frac{\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)}{\lim_{n \rightarrow \infty} 2} = \frac{1}{2}$$

Thus, the answer is $\frac{1}{2}$

Example 3. Find the limit of the numerical sequence whose general term is $x_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \dots + (-1)^{n-1} \frac{1}{2^{n-1}}$.

Solution: The sum forming $\{x_n\}$ is a finite geometric progression with first term $b_1=1$ and common ratio $q=-\frac{1}{2}$.

$$x_n = \frac{1 - \left(-\frac{1}{2}\right)^n}{1 + \frac{1}{2}} = \frac{2}{3} - \frac{2}{3} \left(-\frac{1}{2}\right)^n.$$

Then

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left[\frac{2}{3} - \frac{2}{3} \left(-\frac{1}{2}\right)^n \right] = \lim_{n \rightarrow \infty} \frac{2}{3} - \frac{2}{3} \lim_{n \rightarrow \infty} \frac{(-1)^n}{2^n} = \frac{2}{3}$$

Thus, the answer is $\frac{2}{3}$

Example 4. The n -th term (general term) of the numerical sequence is given by:

$$x_n = 0, \underbrace{333\dots3}_{n \text{ times}}$$

find $\lim_{n \rightarrow \infty} x_n$.

Solution: We write x_n in the following form:

$$x_n = 0,3 + 0,03 + 0,003 + \dots + \underbrace{0,000\dots3}_{(n-1) \text{ times}}$$

This sum is a finite geometric progression with first term $b_1=0.3$, common ratio $q=\frac{1}{10}$, and n -th term $b_n=0.\underbrace{000\dots0}_{(n-1) \text{ times}}3$:

$$x_n = \frac{0,3 \left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}} = \frac{1}{3} \left(1 - \frac{1}{10^n}\right)$$

Then,

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \frac{1}{3} \left(1 - \frac{1}{10^n}\right) = \lim_{n \rightarrow \infty} \frac{1}{3} - \frac{1}{3} \lim_{n \rightarrow \infty} \frac{1}{10^n} = \frac{1}{3}$$

Thus, the answer is $\frac{1}{3}$.

5-misol. Find the limit of the sequence $\{S_n\}$ defined by the sum

$$S_n = q + 2q^2 + 3q^3 + \dots + nq^n, (|q| < 1)$$

Solution: We multiply the given sum by q and then subtract S_n from it:

$$\begin{aligned} qS_n - S_n &= q^2 + 2q^3 + 3q^4 + \dots + nq^{n+1} - (q + 2q^2 + 3q^3 + \dots + nq^n) = \\ &= -(q + q^2 + q^3 + \dots + q^n) + nq^{n+1}. \end{aligned}$$

On the right-hand side, there is the sum of a finite geometric progression with first term $b_1 = q$, common ratio q , and n -th term q^n :

$$\sigma_n = \frac{q(1 - q^n)}{1 - q} = \frac{q - q^{n+1}}{1 - q}, (|q| < 1)$$

Then

$$(q-1)S_n = \frac{q - q^{n+1}}{q-1} + nq^{n+1} \Rightarrow S_n = \frac{nq^{n+1}}{q-1} + \frac{q - q^{n+1}}{(q-1)^2}.$$

According to the condition $|q| < 1$, we have:

$$\lim_{n \rightarrow \infty} nq^{n+1} = 0, \lim_{n \rightarrow \infty} q^{n+1} = 0$$

Then we obtain

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[\frac{nq^{n+1}}{q-1} + \frac{q - q^{n+1}}{(q-1)^2} \right] = \frac{q}{(q-1)^2}, (|q| < 1).$$

Thus, the answer is $\frac{q}{(q-1)^2}$.

Analysis and results.

It is known that various methods are used in calculating numerical sequences; however, the applications of the theory of progressions in solving this problem have not been sufficiently revealed. The applications of this theory have been demonstrated through concrete examples, and reliable results have been obtained.

Conclusion and suggestions.

It is no secret that in calculating limits of numerical sequences, the use of remarkable limits and important limit theorems is common. In this work, it is shown that many limits can also be easily calculated using the theory of progressions. We are confident that this method can be further applied to relevant limit problems in the future to obtain results.

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